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The m-step competition graph of a digraph ☆

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Abstract

The competition graph of a digraph was introduced by Cohen in 1968 associated with the study of ecosystems. Since then, the competition graph has been widely studied and many variations have been introduced. In this paper, we define and study the m-step competition graph of a digraph which is another generalization of competition graph.

Keywords: Competition graph; Competition number; m-step competition graph; m-step competition number

1. Introduction

Cohen [12] introduced the notion of competition graph in connection with a problem in ecology in 1968. The *competition graph* of a digraph D, denoted by $C(D)$, has the same set of vertices as D and an edge between vertices x and y if and only if there is a vertex z in D such that (x,z) and (y,z) are arcs of D (for all undefined graph theory terminology, see [3]). Since the notion of competition graph was introduced, there has been a very large literature on competition graphs. For surveys of the literature of competition graphs, see [22,23,29,41]. In addition to ecology, their various applications include applications to channel assignments, coding, and modeling of complex economic and energy systems (see [34]). There have also been introduced a variety of generalizations of the notion of competition graph, including the common enemy

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graph (sometimes called the resource graph) in [30,40], the competition-common enemy graph (sometimes called the competition-resource graph) in [2,17,21,24,25,36,37], the niche graph in [4,5,7,16,38,39], the p-competition graph in [20,26–28], and the competition multigraph in [1]. In this paper, we introduce yet another such generalization, the m-step competition graph, and obtain results about m-step competition graphs analogous to the well-known results about ordinary competition graphs.

Given a digraph D and a positive integer m, we define the *m-step digraph* D^m of D as follows: $V(D^m)=V(D)$ and there exists an arc (u,v) in D^m if and only if there exists a directed walk of length m from u to v. If there is a directed walk of length m from a vertex x to a vertex y in D, we call y an *m-step prey* of x, and if a vertex w is an m-step prey of both vertices u and v, then we say that w is an *m-step common prey* of u and v. The *m-step competition graph* of D, denoted by $C^m(D)$, has the same vertex set as D and an edge between vertices x and y if and only if x and y have an m-step common prey in D. Note that $C^1(D)$ is the ordinary competition graph of D, and 'directed walk' in the definition of m-step prey can be replaced by 'directed path' for an acyclic digraph D. From the definition of $C^m(D)$ and D^m, the following proposition immediately follows.

Proposition 1. *For any digraph D (possibly with loops) and a positive integer m,*

$$C^m(D)=C(D^m).$$

The concept of m-step digraph and m-step graph are not new, and some asymtotic behavior of D^m is well known (see [6,15]). Moreover some researchers use the concept of 2-step graph to study the competition graphs [31,32]. This motivated us to study the competition graph of D^m (i.e. m-step competition graph of D by the above Proposition). In this paper, Section 2 characterizes m-step competition graphs of digraphs and shows that any spiked n-cycle ($n\geqslant 4$) is not the m-step competition graph of a digraph for any integer $m\geqslant 2$ while any path is a 2-step competition graph of a digraph. Sections 3 and 4 study the m-step competition graph of an acyclic digraph. Section 3 introduces m-step competition numbers, a generalization of the competition numbers introduced by Roberts [35]. Section 4 computes the 2-step competition numbers of paths and cycles. Finally Section 5 proposes some open questions.

2. *m*-step competition graphs

A graph is called an *m-step competition graph* if it is the m-step competition graph of a digraph. In this section, we characterize the m-step competition graphs.

For the two-element Boolean algebra $\mathscr{B}=\{0,1\}$, $\mathscr{B}_n$ denotes the set of all $n\times n$ (Boolean) matrices over $\mathscr{B}$. Under the Boolean operations, we can define matrix addition and multiplication in $\mathscr{B}_n$. Let D be a digraph with vertex set $\{v_1,v_2,\dots,v_n\}$, and $A=(a_{ij})$ be the (Boolean) adjacency matrix of D such that

$$a_{ij}=\begin{cases}1 & \text{if } (v_i,v_j) \text{ is an arc of } D,\\ 0 & \text{otherwise.}\end{cases}$$

Notice that for a positive integer m, the (Boolean) mth power $A^m=(b_{ij})$ of A is a Boolean matrix such that b_{ij} is one if and only if there is a directed walk of length m from v_i to v_j in D. Thus two rows i and i' of A^m have non-zero entry in the jth column if and only if vertex v_j is an m-step common prey of vertices v_i and $v_{i'}$ in D.

A graph G is called the *row graph* G of a matrix A if the rows of A are the vertices of G, and two vertices are adjacent in G if and only if their corresponding rows have a non-zero entry in the same column of A. This notion was studied by Greenberg et al. [18].

Proposition 2. *A graph G with n vertices is an m-step competition graph if and only if there is a Boolean matrix A in $\mathscr{B}_n$ such that G is the row graph of A^m.*

The *edge clique covers* of a graph G are collections of cliques that include all the edges of G. The *edge clique cover number* of a graph G, denoted by $\theta_e(G)$, is the smallest number of cliques in an edge clique cover of G. When A is the adjacency matrix of D, the columns of A^m determine a clique in $C^m(D)$ or consist entirely of zeros (see [18]). Thus the following corollary is true. Notice that one of the theorems of Dutton and Brigham [14] shows that a graph is a (1-step) competition graph if and only if its edge clique cover number is less than or equal to the number of its vertices. Hence, the converse of the following corollary is true when $m=1$. However, Theorem 5 shows that the converse is not necessarily true when $m\geqslant 2$.

Corollary 3. *If G is an m-step competition graph, then the edge clique cover number of G is less than or equal to the number of vertices of G.*

For a Boolean matrix $A\in\mathscr{B}_n$, the *Boolean rank* of A is the smallest integer t such that $A=BC$, where B and C are $n\times t$ and $t\times n$ Boolean matrices respectively (the Boolean rank of a zero matrix is 0) [19]. For $A=(a_{ij})$ and $B=(b_{ij})$ in $\mathscr{B}_n$, we say that A *dominates* B if $b_{ij}\leqslant a_{ij}$ for any i and j. We write $B\leqslant A$ if A dominates B.

Proposition 4 (Cho [10]). *Let the Boolean rank of $A\in\mathscr{B}_n$ be n and $A=BC$ $(B,C\in\mathscr{B}_n)$. Then the Boolean rank of B and C are n, and B and C dominate permutation matrices.*

A non-permutation matrix $A\in\mathscr{B}_n$ is called a *prime Boolean matrix* provided that B or C is a permutation matrix whenever $A=BC$ $(B,C\in\mathscr{B}_n)$ [8]. For a prime Boolean matrix $A\in\mathscr{B}_n$, it is known that the Boolean rank of any square factor of A is n. Thus, every square factor of a prime Boolean matrix dominates a permutation matrix from the above Proposition. A *spiked n-cycle* is a connected graph such that removal of all pendant vertices yields a n-cycle.

Theorem 5. *A spiked n-cycle* ($n \geqslant 4$) *is not an m-step competition graph for any* $m \geqslant 2$.

Proof. By contradiction. Suppose a spiked n-cycle G is the m-step competition graph of a digraph D. Let p be the number of pendant vertices of G and let $A \in \mathscr{B}_{n+p}$ be the adjacency matrix of D. Then for some permutation matrices P and Q,

$$F = PA^m Q = \begin{pmatrix} I_p & O \\ B & C \end{pmatrix}, \tag{1}$$

where I_p is the identity matrix of order p, O is an $p \times n$ zero matrix, B is the $n \times p$ matrix with the property that each column of B contains at most (in fact, exactly one) one 1, and C is the square matrix of order n equal to

$$C = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & 0 & \ddots & \vdots & 0 \\ 0 & 1 & 1 & \ddots & 0 & \vdots \\ 0 & 0 & 1 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 & 1 \end{pmatrix}.$$

Since C is a prime Boolean matrix, the Boolean rank of C is n, and so the Boolean rank of F is $n+p$. Thus the Boolean ranks of PA^k and $A^l Q$ are $n+p$ for any positive integers k and l satisfying $k+l=m$, and there exist a permutation matrix dominated by PA^k and a permutation matrix dominated by $A^l Q$ by Proposition 4. Let R be a permutation matrix dominated by PA^k. Note that $I_{p+n} \leqslant PA^k R^{-1}$. Let

$$X = PA^k R^{-1} = \begin{pmatrix} S & T \\ U & V \end{pmatrix}$$

and

$$Y = RA^l Q = \begin{pmatrix} S' & T' \\ U' & V' \end{pmatrix},$$

where S and S' are square matrics of order p, T and T' are $p \times n$ matrices, U and U' are $n \times p$ matrices, and V and V' are square matrices of order n. Then

$$F = XY = \begin{pmatrix} SS' + TU' & ST' + TV' \\ US' + VU' & UT' + VV' \end{pmatrix}.$$

Thus $ST' + TV' = O$ and so ST' and TV' are zero matrices. Since $I_p \leqslant S$, T must be a zero matrix, and we have $VV' = C$. Note that either V or V' is a permutation matrix since C is a prime Boolean matrix. Since the Boolean rank of C is n, V' dominates a permutation matrix by Proposition 4. Thus T is also a zero matrix. Note that $S = S' = I_p$ since $SS' = I_p$ and $I_p \leqslant S$. Notice that each column of U and U' contains at most one 1 since V dominates a permutation matrix, and $U + VU' = B$.

Suppose m is an even number and let $k=l=m/2$. Then X and Y are permutation equivalent. If V is a permutation matrix, then V' is permutation equivalent to C (in fact, $V'=C$ for $V=I_p$). Thus, there are columns c_1', c_2', c_3' of Y with column sum two such that the inner product of c_i' and c_{i+1}' is one for $i=1,2$, and the inner product of c_1' and c_3' is zero. Since X is permutation equivalent to Y, there exist columns c_1, c_2, c_3 of X where c_i corresponds to c_i' for $i=1,2,3$. But, these columns must occur in

$$\begin{pmatrix} S \\ U \end{pmatrix}$$

since the columns in this matrix are the only possible columns with column sum two. Note that each column can have at most two 1's, one in S and one in U. Since $S'=I_p$ and the inner product of c_i' and c_{i+1}' is 1, c_i must have 1 in the same row for $i=1,2,3$, which contradicts to the fact that the inner product of c_1 and c_3 is 0. We apply a similar argument to reach a contradiction in case where V' is a permutation matrix.

Suppose m ($m\geqslant 2$) is an odd number. Then we let $k=1$ and $l=m-1$. Then $m-1$ is even and Y cannot have the form (1) by the above argument. Thus V' must be a permutation matrix and so V is permutation equivalent to C. Then there exist vertices v_1, v_2, v_3, v_4, x, y, z such that (v_1,x), (v_2,x), (v_2,y), (v_3,y), (v_3,z), (v_4,z) are arcs of D. Moreover, since $I_{p+n}\leqslant X$, there exist a directed path of length $m-2$ from each vertex of D. Thus v_i and v_{i+1} have a common $(m-1)$-step prey in D for $i=1,2,3$. Let v_1, v_2, v_3, v_4 corresponds to rows i_1, i_2, i_3, i_4, respectively. Then there exist three columns c_1'', c_2'', c_3'' in Y such that c_j'' has 1's in rows i_j and i_{j+1} for $j=1,2,3$. These columns belong to

$$\begin{pmatrix} S' \\ U' \end{pmatrix}$$

since their column sums are two. Since $S'=I_p$ and the row sums of rows i_2 and i_3 are at least two, the rows belong to $[U',V']$. Since the inner product of rows i_2 and i_3 is not zero, there must be columns in U' with column sum at least two, which contradicts to the fact that U' has at most one 1 in each column. □

Note that Theorem 5 is not true for $m=1$. For, the edge clique cover number of a spiked cycle G is the same as the number of its vertices and so G is a 1-step competition graph by the theorem of Dutton and Brigham [14] mentioned above.

Theorem 6. *For $n\geqslant 3$, a path P_n of order n is a 2-step competition graph.*

Proof. Let $l=\lceil n/2\rceil$. We take a Boolean matrix A of order n in the form

$$\begin{pmatrix} O_1 & I_{l-1} \\ X & O_2 \end{pmatrix},$$

where O_1 and O_2 are zero matrices with size $l-1$ by $n-l+1$ and size $n-l+1$ by $l-1$, respectively, I_{l-1} is the identity matrix of order $l-1$, and X is a square matrix of order $n-l+1$ with $(1,1)$-entry equal to one, $(i,i-1)$-entries and (i,i)-entries equal to one for $i \geqslant 2$, and the remaining entries equal to zero. Then it is easy to check that for some permutation matrices P and Q of order n,

$$PA^2Q = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ & 1 & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ & & & & 1 & 1 \end{pmatrix},$$

where the unspecified entries are zero. Since the row graph of A^2 is P_n, the theorem follows from Proposition 2. $\square$

3. *m*-step competition numbers

In studying the competition graphs of *acyclic* digraphs, Roberts [35] observed that adding sufficiently many isolated vertices to an arbitrary graph G makes it into the competition graph of some acyclic digraph. The smallest such number of isolated vertices was called the *competition number* of G and denoted by $k(G)$. Much of the study of competition graphs of acyclic digraph has been focused on competition numbers, since the characterization of competition graphs of acyclic digraphs reduces to the question of computing the competition number of an arbitrary graph. We shall use the notation I_r for the graph consisting of r vertices and no edges, and $G \cup I_r$ for the graph consisting of the disjoint union of G and I_r. Analogous to the well-known results for competition graphs, we have the following.

Proposition 7. *Given a graph G and an integer $m \geqslant 1$, there exists a non-negative integer r such that $G \cup I_r$ is the m-step competition graph of an acyclic digraph.*

Proof. Construct a digraph D whose vertices consist of G plus m isolated vertices $v_\alpha^1, v_\alpha^2, \ldots, v_\alpha^m$ for each edge $\alpha = xy$ in G. We let

$$A(D) = \bigcup_{\alpha = xy} \left[\{(x, v_\alpha^1), (y, v_\alpha^1)\} \cup \bigcup_{i=1}^{m-1} \{(v_\alpha^i, v_\alpha^{i+1})\} \right].$$

Then clearly $G \cup I_r = C^m(D)$ for $r = m|E(G)|$. $\square$

Proposition 7 naturally leads us to define the *m-step competition number* $k^{(m)}(G)$ of G, which is the smallest number k such that G together with k isolated vertices is the m-step competition graph of an acyclic digraph. This notion is analogous to the competition number of Roberts [35], the double competition number of Scott [36], the p-competition number of Kim et al. [26], the niche number of Cable et al. [7], and the multicompetition number of Anderson et al. [1]. The following proposition shows that the m-step competition number of a graph G is greater than or equal to the competition number of G.

Proposition 8. *For any graph G and a positive integer m,*

$$k(G) \leqslant k^{(m)}(G).$$

Proof. Let $k = k^{(m)}(G)$. Then there exists an acyclic digraph D such that $C^m(D) = G \cup I_k$. By Proposition 1, $C^m(D) = C(D^m)$. The digraph D^m is clearly acyclic and, from the definition of $k(G)$, it follows that $k(G) \leqslant k$. □

Let D be an acyclic digraph with n vertices. An *acyclic labeling* of the vertex set $V(D)$ of D is a labeling of $V(D)$ using the set $\{v_1, v_2, \ldots, v_n\}$ so that $i < j$ holds whenever there is an arc (v_i, v_j) in D. An acyclic digraph is said to be *acyclically labeled* if its vertices are acyclically labeled.

Opsut [33] showed that for any graph G without isolated vertices,

$$\theta_e(G) - |V(G)| + 2 \leqslant k(G) \leqslant \theta_e(G).$$

The following proposition includes the above result as a special case where $m = 1$.

Proposition 9. *Let G be a graph without isolated vertices. Then*

$$\max\{m, \theta_e(G) - |V(G)| + m + 1\} \leqslant k^{(m)}(G) \leqslant m\theta_e(G).$$

Proof. Suppose that G has n vertices and $\theta_e(G) = l$. Let $\mathscr{C} = \{C_1, \ldots, C_l\}$ be an edge clique cover of size l. We construct a digraph F as follows:

$$V(F) = V(G) \cup \{a_{11}, a_{12}, \ldots, a_{1m}, a_{21}, a_{22}, \ldots, a_{2m}, \ldots, a_{l1}, a_{l2}, \ldots, a_{lm}\}$$

$$A(F) = \bigcup_{i=1}^{l} \{(v, a_{i1}), (a_{i1}, a_{i2}), \ldots, (a_{i,m-1}, a_{im}) \mid v \in V(C_i)\}.$$

Then it can easily be checked that F is acyclic and $C^m(F)$ is $G \cup I_{lm}$. Thus $k^{(m)}(G) \leqslant m\theta_e(G)$.

Now let D be an acyclic digraph such that $C^m(D)$ is G together with $k = k^{(m)}(G)$ isolated vertices. We can give an acyclic labeling of $V(D)$ as $v_1, v_2, \ldots, v_{n+k}$ so that $v_{n+1}, v_{n+2}, \ldots, v_{n+k}$ are the k added isolated vertices. Then $v_1, v_2, \ldots, v_{m+1}$ cannot be used as an m-step prey. On the other hand, since two distinct cliques in $\mathscr{C}$ should prey on different m-step common prey, there should be at least $\theta_e(G)$ distinct vertices used as m-step common prey. Therefore,

$$k^{(m)}(G) \geqslant \theta_e(G) - |V(G)| + m + 1.$$

To complete the proof of the first inequality, we note that v_n is adjacent to at least one vertex of G since G has no isolated vertices. Therefore v_n should have an m-step common prey in D. Since any vertex having a label less than n cannot be an out-neighbor of v_n in D, it follows that $k^{(m)}(G) \geqslant m$. □

The following corollary is an immediate consequence of Proposition 9.

Corollary 10. $k^{(m)}(K_n) = m$ *for any complete graph K_n with $n \geqslant 2$.*

4. 2-step competition numbers of paths and cycles

Throughout the rest of this paper, we denote by M_n the digraph with

$$V(M_n)=\{v_1,v_2,\ldots,v_{n+2}\}$$

and

$$A(M_n)=\{(v_1,v_3)\}\cup\{(v_i,v_{i+1})\,|\,1\leqslant i\leqslant n+1\}\cup\{(v_i,v_{i+3})\,|\,i \text{ even},\ 2\leqslant i\leqslant n-1\}.$$

Let $l=\lfloor n/2\rfloor$. Then it is not difficult to check that $C^2(M_n)$ is the path $v_{2l}v_{2(l-1)}\ldots v_4v_2v_1v_3\ldots v_{2l-1}v_{2l+1}$ together with two isolated vertices v_{n+1} and v_{n+2} if n is odd, and the path $v_{2l}v_{2(l-1)}\ldots v_4v_2v_1v_3\ldots v_{2l-3}v_{2l-1}$ with two isolated vertices v_{n+1} and v_{n+2} if n is even. That is, the 2-step competition graph of M_n is $P_n\cup I_2$. Thus $k^{(2)}(P_n)\leqslant 2$. From this observation and Proposition 9, the following theorem can be obtained.

Theorem 11. *For any integer* $n\geqslant 2$, $k^{(2)}(P_n)=2$.

Given an acyclic digraph D with n vertices and an acyclic labeling $v_1,v_2,\ldots,v_n$ of $V(D)$, we call an arc (v_i,v_j) in D a *jump-arc* when $i+1<j$.

Lemma 12. *For* $n\geqslant 4$, *let* $k=k^{(2)}(C_n)$. *Let D be an acyclic digraph whose 2-step competition graph is* $C_n\cup I_k$. *Let* $v_1,v_2,\ldots,v_{n+k}$ *be an acyclic labeling of* $V(D)$. *Then* $C^2(D-v_{n+k})$ *is* $P_n\cup I_{k-1}$.

Proof. By the definition of an acyclic labeling, the outdegree of v_{n+k} is zero and, by the definition of the competition number, v_{n+k} should be a 2-step common prey of two vertices of C_n. Thus deleting v_{n+k} from D results in the deletion of edge in $C^2(D)$ joining the two vertices. Hence the 2-step competition graph of $D-v_{n+k}$ is $P_n\cup I_{k-1}$. □

Lemma 13. *Let n be any integer greater than one and D be an acyclic digraph such that* $C^2(D)$ *is* $P_n\cup I_2$. *Let* $v_1,v_2,\ldots,v_{n+2}$ *be an acyclic labeling of* $V(D)$. *Then the following are true*:

(i) $C^2(D-v_{n+2})$ *is* P_{n-1} *with two isolated vertices.*

(ii) *For any i with* $2\leqslant i\leqslant n+1$, *there exists an arc* (v_i,v_{i+1}) *in D.*

Proof. By the definition of an acyclic labeling, v_{n+2} and v_{n+3} do not have a 2-step prey and therefore are isolated vertices in $C^2(D)$. Since v_{n+2} is the only 2-step prey of v_n, v_n is the end vertex of P_n. Thus deleting v_{n+2} from D results in the deletion of the edge incident to v_n in $C^2(D)$. Hence $C^2(D-v_{n+2})$ is P_{n+2} with two isolated vertices v_n and v_{n+1} and (i) follows.

We prove (ii) by inducting on n. When $n=2$, v_4 is a 2-step prey of v_2 and hence arcs (v_2,v_3) and (v_3,v_4) must be in D. Suppose that (ii) holds for $n-1$, $n\geqslant 3$. By (i), $C^2(D-v_{n+2})$ is $P_{n-1}\cup I_2$. Thus by the induction hypothesis, for any i with $2\leqslant i\leqslant n$, arc

(v_i, v_{i+1}) is in $D - v_{n+2}$ and so in D. Since v_{n+2} is a 2-step prey of v_n, arc (v_{n+1}, v_{n+2}) must be in D. Hence (ii) follows. □

Lemma 14. *Let n be any integer greater than one and D be an acyclic digraph such that $C^2(D)$ is $C_n \cup I_3$. Let $v_1, v_2, \ldots, v_{n+3}$ be an acyclic labeling of $V(D)$. Then the following are true*:

(i) *For any i, $2 \leqslant i \leqslant n+1$, there exists an arc (v_i, v_{i+1}) in D.*

(ii) *If there is an incoming jump-arc toward v_j, then there is no outgoing jump-arc from v_j.*

(iii) *If $v_i v_j$ $(i < j)$ is an edge in $C^2(D)$, then either (v_i, v_{j+1}) or (v_{i+1}, v_{j+2}) is a jump-arc of D.*

(iv) *For any i, $3 \leqslant i \leqslant n$, either (v_{j+1}, v_i) or (v_j, v_{i-1}) is a jump-arc of D for some $j < i-2$.*

(v) *For any i, $i \geqslant 3$, v_i cannot be ends of more than one jump-arc.*

Proof. By Lemma 12, $C^2(D - v_{n+3})$ is $P_n \cup I_2$ and therefore, by Lemma 13(ii), $D - v_{n+3}$ and D contain arcs (v_i, v_{i+1}) for i, $2 \leqslant i \leqslant n+1$.

We prove (ii) by contradiction. Suppose that $(v_i,\ v_j)$ and $(v_j,\ v_l)$ are jump-arcs in D. Then v_l is a 2-step common prey of v_i, v_{j-1}, and v_{l-2}. Since $i < j-1$ and $j < l-1$, the three vertices v_i, v_{j-1} and v_{l-2} are distinct and form a cycle C_3 in $C^2(D)$, which is a contradiction. Hence (ii) follows.

Suppose that $v_i v_j$ $(i < j)$ is an edge in $C^2(D)$. Then there exists l, $l \geqslant j+2$, such that v_l is a 2-step common prey of v_i and v_j. In fact $l = j+2$, for otherwise v_l is a 2-step common prey of three distinct vertices v_{l-2}, v_j and v_i. Since v_{j+2} is a 2-step prey of v_i, arcs (v_i, v_p) and (v_p, v_{j+2}) are in D for some p. By (ii), either $p = i+1$ or $p = j+1$. Hence (iii) follows.

We prove (iv). Since v_1, v_2, v_3 cannot be used as 2-step common prey and there are n edges no three of which form a triangle, v_i is used as a 2-step common prey of v_{i-2} and v_j for some $j < i-2$. By (iii), either (v_j, v_{i-1}) or (v_{j+1}, v_i) is an arc of D and (iv) follows.

We prove (v) by contradiction. Suppose that v_i is ends of two distinct jump-arcs (v_r, v_s) and (v_t, v_u) for some $r, s, t, u \in \{1, 2, \ldots, n+2\}$ with $s \geqslant r+2$ and $u \geqslant t+2$. By (ii), either $i = r = t$ and $s \neq u$ or $i = s = u$ and $r \neq t$. If $i = r = t$, then v_i has three 2-step prey v_{i+2}, v_{s+1} and v_{u+1}. But, by (iv), there exists a jump-arc (v_j, v_{i+1}) or (v_{j+1}, v_{i+2}) for some $j < i$. Then v_i is adjacent to three distinct vertices v_j, v_{s-1}, and v_{r-1}, which is a contradiction. If $i = s = u$, then v_i is a 2-step common prey of three distinct vertices v_{i-2}, v_{r-1}, and v_{s-1}, which is a contradiction. □

By Corollary 10, $k^{(2)}(C_3) = 2$. The following theorem gives the 2-step competition number of a cycle of length greater than or equal to 4.

Theorem 15. *For $n \geqslant 4$, $k^{(2)}(C_n) = 4$.*

Proof. By Proposition 9, $k^{(2)}(C_n) > 2$. Let D be the digraph obtained from M_n by adding two vertices v_{n+3}, v_{n+4}, arc (v_{n+3}, v_{n+4}), and then adding arcs (v_n, v_{n+3}), (v_n, v_{n+4}) if n is even; arcs (v_{n-1}, v_{n+3}), (v_{n+1}, v_{n+4}) if n is odd. Then it can easily be checked that D is acyclic and $C^2(D)$ is $C_n \cup I_4$. Thus $k^{(2)}(C_n) = 3$ or 4.

We will show by contradiction that $k^{(2)}(C_n) \neq 3$. Assume that $k^{(2)}(C_n) = 3$. Let D be an acyclic digraph such that $C^2(D)$ is $C_n \cup I_3$ and the 2-step competition graph of subdigraph of D is not $C_n \cup I_3$. There is an acyclic labeling $v_1, v_2, \ldots, v_{n+3}$ of $V(D)$. Then, by Lemma 14(i), D contains arc (v_i, v_{i+1}) for $i = 2, 3, \ldots, n+1$. Now we claim the following:

Claim. *D has arc (v_{n+1}, v_{n+3}), but D does not have arc (v_{n+2}, v_{n+3}).*

Proof of Claim. Since v_n has only one 2-step prey v_{n+2} in $D - v_{n+3}$, v_n is an end vertex of P_n in $C^2(D - v_{n+3})$, and so v_{n+3} is a 2-step common prey of v_n and the other end vertex of P_n. Since v_{n+3} is a 2-step prey of v_n, either (v_{n+2}, v_{n+3}) or (v_{n+1}, v_{n+3}) is in D. If (v_{n+2}, v_{n+3}) is in D, then v_{n+3} is a 2-step prey of v_{n+1}. Since v_{n+3} is also a 2-step prey of v_n, v_n and v_{n+1} are joined in $C^2(D)$, contradicting the fact that v_{n+1} is an isolated vertex in $C^2(D)$. Thus (v_{n+1}, v_{n+3}) is in D and the claim follows.

Note that each of the n vertices $v_4, \ldots, v_{n+2}, v_{n+3}$ must be the only 2-step common prey of exactly two vertices. Then there is no jump-arc (v_l, v_{n+1}); otherwise vertices v_l and v_n share two common 2-step prey v_{n+2}, v_{n+3}. Thus we must have jump-arcs (v_r, v_{n+2}) and (v_s, v_{n+3}) for some distinct r, $s \in \{2, \ldots, n\}$. Note that the ends of each edge of C_n are connected to their common 2-step prey by a directed path of length 2 exactly one of whose arcs is a jump-arc and that any jump-arc is associated with at most 2 paths of length 2 to prey. Also note that each of arcs $(v_r, v_{n+2}), (v_s, v_{n+3})$ and jump-arcs outgoing from v_1 contributes one edge to $C^2(D)$. We consider the following two cases.

Case 1: *(v_1, v_2) is not an arc of D.* Then there are exactly two jump-arcs (v_1, v_3) and (v_1, v_t) from v_1 for $t \in \{4, \ldots, n\} \setminus \{r, s\}$. Since there cannot be outgoing jump-arc from v_3 by Lemma 14(ii) and v_2 is adjacent to one more vertex other than v_1, there must be exactly one jump-arc (v_2, v_u) for some $u \in \{4, 5, \ldots, n\} \setminus \{r, s, t\}$ by Lemma 14(iii). If $n = 4$, then $t = 4$ and 2 is the only possible choice for r and s, which is a contradiction. Now assume that $n \geqslant 5$. Then arcs $(v_1, v_3), (v_1, v_t), (v_2, v_u), (v_r, v_{n+2})$ and (v_s, v_{n+3}) are used in paths of length 2 to the prey which are $v_4, v_{t+1}, v_{u+1}, v_{n+2}, v_{n+3}$, respectively, giving exactly five edges of $C^2(D)$. By Lemma 14(ii) and (v) and the observation that v_{n+1} has no incoming arc, the jump-arcs other than $(v_r, v_{n+2}), (v_s, v_{n+3}), (v_1, v_3)$, and (v_1, v_t) have ends in $\{v_4, v_5, \ldots, v_n\} \setminus \{v_r, v_s, v_t\}$, a set of size $n - 6$. Hence there are at most $(n-6)/2$ other jump-arcs, each contributing at most 2 paths of length 2 so these other jump-arcs are associated with at most $n - 6$ edges of C_n. Along with the five edges noted above we get only $n - 1$ edges for C_n, which is a contradiction.

Case 2: *(v_1, v_2) is an arc of D.* Suppose that there is a jump-arc (v_2, v_4) in D. Then either $2 \in \{r, s\}$ and there is no jump-arc from v_1, or $2 \notin \{r, s\}$ and there is exactly one jump-arc (v_1, v_t) outgoing from v_1 for some $t \in \{5, 6, \ldots, n\}$. Since (v_2, v_4) is in

D, there cannot be jump-arc outgoing from v_3. If $n=4$, then 2 is the only possible choice for r and s, which is a contradiction. Thus we may assume $n \geqslant 5$. Note that the jump-arc (v_2, v_4) contributes two edges to $C^2(D)$. Suppose that $2 \in \{r, s\}$ and there is no jump-arc from v_1. Then the jump-arcs other than $(v_r, v_{n+2}), (v_s, v_{n+3})$, and (v_2, v_4) must have ends in $\{v_5, \ldots, v_n\} \setminus \{v_u\}$ where $u \in \{r, s\} \setminus \{2\}$, and we have at most $n-5+4=n-1$ edges for $C^2(D)$, which is a contradiction. Suppose that $2 \notin \{r, s\}$ and there is exactly one jump-arc (v_1, v_t) outgoing from v_1 for some $t \in \{5, 6, \ldots, n\}$. If $n=5$, then $t=5$ and v_6 is the only 2-step prey of v_4 since a jump-arc from v_4 or v_5 is not allowed, which is a contradiction. Suppose that $n \geqslant 6$. Then the jump-arcs other than (v_r, v_{n+2}), $(v_s, v_{n+3}), (v_1, v_t)$, and (v_2, v_4) must have ends in $\{v_5, \ldots, v_n\} \setminus \{v_r, v_s, v_t\}$, a set of size $n-7$ (r, s, t must be distinct) and we get at most $n-7+5=n-2$ edges for $C^2(D)$, which is a contradiction.

Suppose that no jump-arc (v_2, v_4). Then there must be jump-arc (v_1, v_3) and jump-arc (v_2, v_u) for some $u \in \{5, \ldots, n\} \setminus \{r, s\}$ and these are only jump-arcs outgoing from v_1 and v_2. If $n=4$, then u cannot exist. If $n=5$, then $u=5$ and v_6 is the only possible 2-step prey of v_4, which is a contradiction. Assume that $n \geqslant 6$. Note that the jump-arc (v_2, v_u) contributes two edges to $C^2(D)$. The jump-arcs other than (v_r, v_{n+2}), (v_s, v_{n+3}), (v_1, v_3), and (v_2, v_u) must have ends in $\{v_4, v_5, \ldots, v_n\} \setminus \{v_r, v_s, v_u\}$, a set of size $n-6$ and we have at most $n-6+5=n-1$ edges for $C^2(D)$, which is a contradiction.

□

5. Closing remarks

Theorem 6 shows that a path of length at least 3 is a 2-step competition graph. But, we do not know that it is an m-step competition graph for $m \geqslant 3$.

Proposition 8 shows that $k(G) \leqslant k^{(m)}(G)$ for any positive integer m. We note that the lower bound of that inequality is not achieved by any of the complete graphs, paths, and cycles whose 2-step competition numbers are found in this paper. In fact, we have not found any graph G satisfying $k(G)=k^{(2)}(G)$ and propose a problem to prove or disprove that $k^{(m)}(G) > k(G)$ for any integer $m \geqslant 2$.

Cho et al. [10,11] characterized the trees whose 2-step competition numbers are two. However, computing the 2-step competition number of an arbitrary tree does not appear easy. One interesting problem on 2-step competition numbers of trees would be to see whether or not the 2-step competition number of a tree can be arbitrarily large.

It will be worthwhile to extend our results to find formulas for m-step competition numbers of paths and cycles for an integer $m \geqslant 3$.

6. For further reading

The following references are also of interest to the reader: [9,13].

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